Liouville Equation, Painlevé Property and Bäcklund Transformation

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We demonstrate that a Bäcklund transformation for the Liouville equation can be obtained in a straightforward manner from the Painlevé property of this equation.

Recently, much attention has been focused to the classification of dynamical systems as integrable and non-integrable ones. When we consider classical mechanics, the Toda lattice is a well known example of an integrable system [1]. There are N first integrals in involution and there is Lax representation. In field theory the Korteweg de Vries equation is an integrable system. This equation can be solved with the help of the inverse scattering transform. Moreover there is an infinite number of conservation laws [2]. In quantum field theory the best known example of an integrable system is the quantum nonlinear Schrödinger equation. This system can be solved with the help of the so-called quantum spectral method or with the help of the Bethe ansatz [3]. However, most dynamical systems are non-integrable. In classical mechanics we find among these non-integrable systems those with chaotic behaviour.

Now it is desirable to have a simple approach for deciding whether a dynamical system is integrable or not. In classical mechanics the so-called Painlevé property serves to distinguish between integrable and non-integrable systems. A necessary condition for an ordinary differential equation to have the Painlevé property is that there be a Laurent expansion which represents the general solution in a deleted neighbourhood of a pole (compare [4] and references therein). Recently, Weiss *et al.* [5] have introduced the Painlevé property for partial differential equations. They applied the method to the soliton equations (like Korteweg de Vries, Kadomtsev-Petviashvili) and found, in a remarkably straightforward manner, the well known Bäcklund transformations.

In the present note we apply the method of Weiss *et al.* [5] to the Liouville equation. We show that a Bäcklund transformation can be obtained in a straightforward manner.

It is well known that the Liouville equation

$$u_{xt} = \exp(u) \tag{1}$$

can be reduced with the help of the Bäcklund transformation

$$u'_x = u_x + \beta \exp((1/2)(u + u')),$$
$$u'_t = -u_t - (2/\beta) \exp(1/2)(u - u')) \tag{2}$$

to the linear equation

$$u'_{xt} = 0. \tag{3}$$

Thus, insertion of the general solution

$$u'(x, t) = f_1(x) + f_2(t) \tag{4}$$

of (3) into the Bäcklund transformation (2) and subsequent integration produces the general solution of Liouville's equation.

Now the method of Weiss *et al.* [5] cannot be applied directly to (1), so we must perform the transformation $v = \exp(u)$. Then we obtain the equation

$$v v_{xt} - v_x v_t = v^3. \tag{5}$$

Equation (3) takes the form

$$v' v'_{xt} - v'_x v'_t = 0, \tag{6}$$

where $v' = \exp(u')$. From (4) we find that the solution of (6) is given by

$$v'(x, t) = \exp(f_1(x) + f_2(t)). \tag{7}$$

In the technique described by Weiss *et al.* [5] the quantities v, x, and t are considered in the complex domain. For the sake of simplicity we do not change our notation. For the field v we make the series ansatz

$$v(x, t) = \Phi^\alpha(x, t) \sum_{j=0}^{\infty} v_j(x, t) \Phi^j(x, t). \tag{8}$$

If α is an integer, and if it is possible to cut off this series expansion at a certain integer, say n ($n < \infty$), and moreover the equations for the functions

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Φ, v_0, $v_1, \ldots, v_n$ are compatible, then we obtain a Bäcklund transformation.

Let us now perform the calculation step by step for (5). First we determine the dominant behaviour, *i.e.* we determine the exponent α. Inserting the ansatz $v \sim \Phi^\alpha v_0$ into (5) and comparing the exponents we find that $\alpha = -2$, and the function v_0 is given by $v_0 = 2\Phi_x \Phi_t$. Next we determine the resonances. The values of j are called resonances where arbitrary functions u_j are introduced into the expansion. Inserting the ansatz (8) with $\alpha = -2$ into (5) we find

$$\begin{aligned}
\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} [(j_2-2)(j_3-3) v_{j_1} v_{j_2} \Phi_t \Phi_x \Phi^{j_1+j_2-6} \\
+ (j_2-2) v_{j_1} v_{j_2x} \Phi_t \Phi^{j_1+j_2-5} \\
+ (j_2-2) v_{j_1} v_{j_2t} \Phi_x \Phi^{j_1+j_2-5} \\
+ (j_2-2) v_{j_1} v_{j_2} \Phi_{xt} \Phi^{j_1+j_2-5} \\
+ v_{j_1} v_{j_2xt} \Phi^{j_1+j_2-4}] \\
- \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} [(j_1-2)(j_2-2) v_{j_1} v_{j_2} \Phi_x \Phi_t \Phi^{j_1+j_2-6} \\
+ (j_1-2) v_{j_1} v_{j_2x} \Phi_t \Phi^{j_1+j_2-5} \\
+ (j_1-2) v_{j_1} v_{j_2t} \Phi_x \Phi^{j_1+j_2-5} \\
+ v_{j_1x} v_{j_2t} \Phi^{j_1+j_2-4}] \\
= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} v_{j_1} v_{j_2} v_{j_3} \Phi^{j_1+j_2+j_3-6}. \qquad (9)
\end{aligned}$$

The resonances are determined from the terms with the factors $\Phi^{j_1+j_2-6}$ and $\Phi^{j_1+j_2+j_3-6}$. We find

$$\begin{aligned}
3 v_m v_0^2 = 6 v_0 v_m \Phi_t \Phi_x + (m-2)(m-3) v_m v_0 \Phi_t \Phi_x \\
+ 2(m-2) v_m v_0 \Phi_t \Phi_x \\
+ 2(m-2) v_0 v_m \Phi_t \Phi_x . \qquad (10)
\end{aligned}$$

Inserting $v_0 = 2\Phi_x \Phi_t$, we find that $m_1 = -1$ and $m_2 = 2$. The value $m_1 = -1$ corresponds to the arbitrary (undefined) singularity manifold ($\Phi = 0$).

Solving (10) we find that

$$\begin{aligned}
j = 0, \quad v_0 = 2\Phi_x \Phi_t, \\
j = 1, \quad v_1 = -2\Phi_{xt}. \qquad (11)
\end{aligned}$$

The compatability condition at $j = 2$ is satisfied identically. If we put $v_j = 0$ for $j \geqq 3$, then we find for $j = 6$ that v_2 must satisfy (5), *i.e.*

$$v_2 v_{2xt} - v_{2x} v_{2t} = v_2^3. \qquad (12)$$

The trivial solution of (12) is given by $v_2 = 0$. Now let us assume that $v_j = 0$ for $j \geqq 2$. Then we find the following overdetermined system of equations for the function Φ:

$$\begin{aligned}
j = 3, \quad \Phi_{xxt} \Phi_x \Phi_{tt} + \Phi_{xtt} \Phi_{xx} \Phi_t \\
= \Phi_{xx} \Phi_{tt} \Phi_{xt} + \Phi_x \Phi_t \Phi_{xxtt}, \qquad (13\text{a})
\end{aligned}$$

$$j = 4, \quad \Phi_{xxtt} \Phi_{xt} = \Phi_{xxt} \Phi_{xtt}. \qquad (13\text{b})$$

The equation at $j = 5$ is satisfied identically. Now the function v is given by

$$v = \Phi^{-2}(v_0 + v_1 \Phi) = 2\Phi^{-2}(\Phi_x \Phi_t - \Phi_{xt} \Phi). \qquad (14)$$

If (13a) and (13b) are compatibel, then (14) defines a Bäcklund transformation (5). We mention that if we insert (14) into (5) we find the function Φ satisfies (13a) and (13b). Equations (13a) and (13b) are compatibel. This means both equations admit the solution

$$\Phi(x, t) = f_1(x) + f_2(t). \qquad (15)$$

Thus (14) defines a Bäcklund transformation and when we insert (15) into (14) we obtain a solution of (5).

The Liouville equation in (1 + 1) dimensions may be written as $u_{xt} = \exp(u)$ or as $u_{xx} - u_{tt} = \exp(u)$ depending on the underlying coordinate system. Performing the transformation $v = \exp(u)$ we find for the second equation

$$v v_{xx} - v_x^2 - v v_{tt} + v_t^2 = v^3. \qquad (16)$$

When we insert the ansatz (8) into (16) we obtain $\alpha = -2$, and the resonances are given by $m_1 = -1$, $m_2 = 2$. Moreover, we have

$$v_0 = 2(\Phi_x^2 - \Phi_t^2), \qquad (17\text{a})$$

$$v_1 = -2(\Phi_{xx} - \Phi_{tt}), \qquad (17\text{b})$$

and $v = \Phi^{-2}(v_0 + v_1 \Phi)$ defines a Bäcklund transformation. The extension to two and three space dimensions is straightforward.

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